MATRIX INVERSES

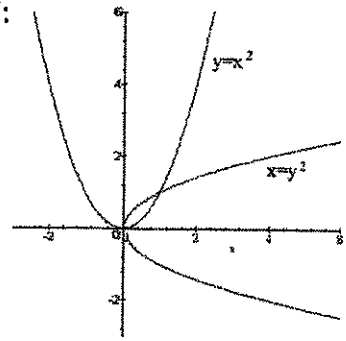
Recall that *inverses* are operations that "undo" each other. You are already familiar with many of these, e.g. . . .

The Additive Inverse _____ EX: $4 + (-4) = 0$

The Multiplicative Inverse _____ EX: $3 \times \frac{1}{3} = 1$

Powers and Roots _____ EX: $5^2 = \sqrt{25}$

Graphs of Functions _____ EX:



A matrix has an inverse iff. . .

A. it is a square matrix

B. the determinant is not equal to zero

$$I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product of a matrix and its inverse is always equal to the identity matrix $I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The following matrices are inverses of each other. Verify by finding their product.

EX $\begin{bmatrix} 7 & 3 \\ 5 & +2 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix} = \begin{bmatrix} -14+15 & 21-21 \\ -10+10 & 15-14 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

EX $\begin{bmatrix} 11 & 3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ -\frac{3}{2} & \frac{11}{4} \end{bmatrix} = \begin{bmatrix} \frac{11}{2}-\frac{9}{2} & -\frac{33}{4}+\frac{33}{4} \\ \frac{6}{2}-\frac{6}{2} & -\frac{18}{4}+\frac{22}{4} \end{bmatrix}$

$\begin{bmatrix} \frac{2}{2} & 0 \\ 0 & \frac{4}{4} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

To find the inverse of a 2 x 2 matrix A. . .

Step 1: Find the determinant of A*.

Step 2: Apply the formula $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Step 3: Distribute the scalar.

*If the determinant = 0, then the inverse is *undefined*.

EX 1 Let $A = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}$

$|A| = (3)(6) - (-5)(-4)$

$18 - 20$

-2

$A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & 4 \\ 5 & 3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -3 & -2 \\ -\frac{5}{2} & -\frac{3}{2} \end{bmatrix}$

EX 2 Let $A = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$

$|A| = (8)(2) - (3)(5)$

$= 16 - 15$

1

$A^{-1} = 1\begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & -5 \\ -3 & 8 \end{bmatrix}$

SOLVING SYSTEMS USING MATRIX INVERSES

In order to solve a matrix equation, we can left-multiply both sides of the equation by the multiplicative inverse.

$$A\ X = B$$

Solve the matrix equation $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} X = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$

$$\underbrace{A^{-1}A}\ X = A^{-1}B$$

$$I\ X = A^{-1}B$$

$$X = A^{-1}B$$

$$|A| = (2)(2) - (3)(1)$$

$$4 - 3$$

$$1$$

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

$$X = A^{-1}B \rightarrow \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 8 & 1 \\ -11 & -1 \end{bmatrix}$$

FIND the inverse of matrix A

LEFT-MULTIPLY THE INVERSE TO MATRIX B

HINT: it's better to distribute $\frac{1}{|A|}$ as the last step

In order to solve a system of equations using a matrix inverse. . .

Step 1: Rewrite the system in matrix form.

Step 2: Left-multiply both sides of the equation by the multiplicative inverse of the matrix of coefficients.

EX 1 $5x+3y=-5$, $7x+5y=-11$

matrix of variables

$$\begin{bmatrix} 5 & 3 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ -11 \end{bmatrix}$$

matrix of coefficients

matrix of constants

$$|A| = (5)(5) - (7)(3)$$

$$25 - 21$$

$$4$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\frac{1}{4}\begin{bmatrix} 5 & -3 \\ -7 & 5 \end{bmatrix}}_{A^{-1}}\underbrace{\begin{bmatrix} -5 \\ -11 \end{bmatrix}}_{B} \rightarrow \frac{1}{4}\begin{bmatrix} 8 \\ -20 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

$\therefore x = 2,\ y = -5$

EX 2 $4x+5y=7$, $-2x-y=7$

$$\begin{bmatrix} 4 & 5 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$|A| = (4)(-1) - (-2)(5)$$

$$-4 + 10$$

$$6$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & -5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} -42 \\ 42 \end{bmatrix} = \begin{bmatrix} -7 \\ 7 \end{bmatrix}$$

$\therefore x = -7,\ y = 7$

Solving Matrix Equations Using an Inverse Name _____

1. How can we determine whether two given matrices are inverses of each other?

If their product is equal to the identity matrix

2. In order for a matrix to have an inverse, what two criteria need to be met?

- the matrix must be square
- the determinant $\neq 0$

Determine whether the given matrices are inverses of each other. Justify your answer.

3. $\begin{bmatrix} 0 & 2 & -1 \\ 5 & 2 & 3 \\ 7 & 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} -1 & -11 & 8 \\ 1 & 7 & -5 \\ 1 & 14 & -10 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. $\begin{bmatrix} -1 & 2 \\ -5 & 9 \end{bmatrix}$ and $\begin{bmatrix} 9 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} -19 & 4 \\ -90 & 19 \end{bmatrix}$

NO!

Solve the following matrix equations using a matrix inverse.

5. $\begin{bmatrix} -5 & -13 \\ 2 & 5 \end{bmatrix} X = \begin{bmatrix} 3 & 1 \\ -4 & 0 \end{bmatrix}$

$A \quad X = B$

$|A| = (-5)(5) - (2)(-13)$

$-25 + 26$

1

$X = \frac{1}{1}\begin{bmatrix} 5 & 13 \\ -2 & -5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -4 & 0 \end{bmatrix}$

$X = \begin{bmatrix} -37 & 5 \\ 14 & -2 \end{bmatrix}$

$\begin{bmatrix} -37 & 5 \\ 14 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} X + \begin{bmatrix} 2 & 7 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ -2 & 3 \end{bmatrix}$

$\begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 4 & -5 \\ 1 & -1 \end{bmatrix}$

$|A| = 8 - 7 = 1$

$X = \frac{1}{1}\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$

7. $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 4 & 0 & 6 \\ 3 & -1 & 5 \end{bmatrix}$

A X B

$\begin{bmatrix} -4 & 2 & -7 \\ 3 & -1 & 5 \end{bmatrix}$

$|A| = 2 - 0 = 2$

$X = \frac{1}{2}\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 & 6 \\ 3 & -1 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -8 & 4 & -14 \\ 6 & -2 & 10 \end{bmatrix}$

8. $\begin{bmatrix} -7 & -9 \\ 4 & 5 \end{bmatrix} X + \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 6 & -6 \end{bmatrix}$

$\begin{bmatrix} 8 & -2 \\ -6 & 1 \end{bmatrix}$

$\begin{bmatrix} -7 & -9 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 2 & -3 \end{bmatrix}$

$X = \frac{1}{1}\begin{bmatrix} 5 & 9 \\ -4 & -7 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 2 & -3 \end{bmatrix}$

$|A| = (-7)(5) - (4)(-9)$

$-35 + 36 = 1$

9. $\begin{bmatrix} -6 & -3 \\ 3 & 1 \end{bmatrix} X = \begin{bmatrix} 9 & 12 & 0 \\ -4 & 5 & -2 \end{bmatrix}$

$\begin{bmatrix} -1 & 9 & -2 \\ -1 & -22 & 4 \end{bmatrix}$

$|A| = (-6)(1) - (3)(-3)$

$-6 + 9$

3

$X = \frac{1}{3}\begin{bmatrix} 1 & 3 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} 9 & 12 & 0 \\ -4 & 5 & -2 \end{bmatrix}$

$X = \frac{1}{3}\begin{bmatrix} -3 & 27 & -6 \\ -3 & -66 & 12 \end{bmatrix}$

Rewrite the system as a matrix equation, then solve using an inverse.

10. $4x + 3y = 2$
$2x + 4y = 6$

$\begin{bmatrix} 4 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$|A| = 16 - 6 = 10$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & -3 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -10 \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

11. $2x - y = 5$
$x - 2y = 1$

$\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$|A| = -4 - (-1) = -3$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$= \frac{1}{-3}\begin{bmatrix} -9 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

12. $3x + 4y = -8$
$-5x + y = -25$

$\begin{bmatrix} 3 & 4 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ -25 \end{bmatrix}$

$|A| = (3)(1) - (-5)(4)$

$3 + 20$

23

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 1 & -4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} -8 \\ -25 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 92 \\ -115 \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$